13.4 Position, Velocity, Acceleration

If ***t = time*** and position is given by

$$\boldsymbol{r}(t) = < x(t), y(t), z(t) >$$

then

$$\boldsymbol{r}'(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h}$$
$$= \frac{\text{change in position}}{\text{change in time}}$$
$$= \text{velocity} = \boldsymbol{v}(t)$$

$$|\boldsymbol{r}'(t)| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed}$$

$$\boldsymbol{r}''(t) = \lim_{h\to 0}\frac{\boldsymbol{r}'(t+h)-\boldsymbol{r}'(t)}{h}$$
$$= \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \text{acceleration} = \boldsymbol{a}(t)$$

Example:

Let t be **time in seconds** and assume the position of an object (with components in **feet**) is given by

$$\boldsymbol{r}(t) = < t, 2e^{-t}, 0 >$$

Compute

1. $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$ and $\boldsymbol{r}''(t)$.
2. $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$ and $\boldsymbol{r}''(0)$.

HUGE application:
Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \quad \text{, so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant.)

If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve differential equations) to find velocity and position.

This is how we can model ALL motion problems!

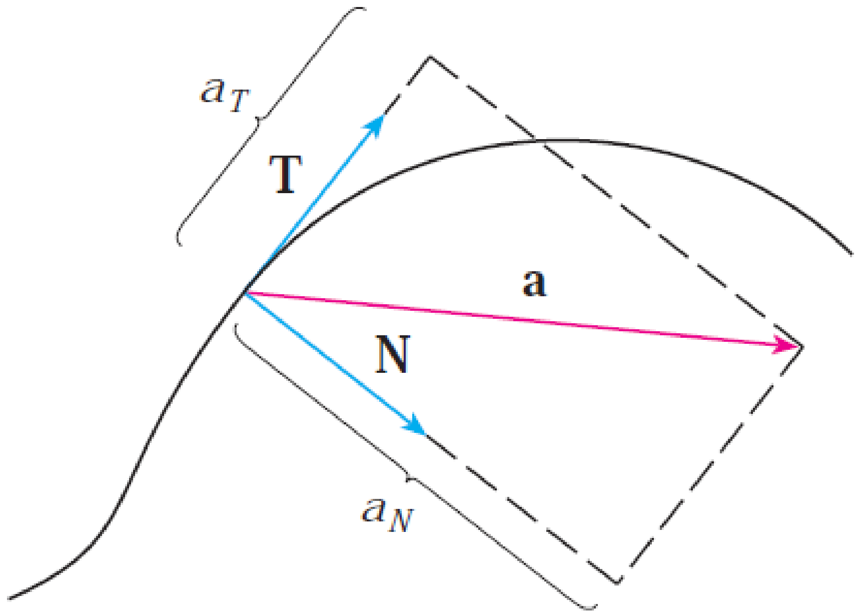
Example:
A ball with mass *m = 0.8* kg is thrown northward into the air with initial speed of 30 m/sec at an angle of 30 degrees with the ground.

A west wind applies a steady force of 4 N on the ball (west to east). If you are standing on level ground, where does the ball land?

Steps (for *all* motion problems):

1. Forces?
2. Get acceleration.
3. Integrate to get $\vec{\boldsymbol{v}}(t)$
   (initial conditions?)
4. Integrate again to get $\vec{\boldsymbol{r}}(t)$
   (initial conditions?)

Measuring and describing acceleration



Recall: $\text{comp}_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b}}$ = length.

We define the tangential and normal components of acceleration by:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$ = tangential

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$ = normal

Note that: $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$

Derivation of interpretation:

Let $\nu(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}\prime(t)}{|\vec{\boldsymbol{r}}'(t)|} = \frac{\vec{\boldsymbol{v}}(t)}{\nu(t)}$ implies $\vec{\boldsymbol{v}} = \nu \vec{\boldsymbol{T}}$.

2. $\kappa(t) = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{T}}'|}{\nu(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa\nu$.

3. $\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \frac{\vec{\boldsymbol{T}}'}{\kappa\nu}$,implies $\vec{\boldsymbol{T}}' = \kappa\nu\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + \nu\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + k\nu^2\vec{\boldsymbol{N}}.$$

Conclusion

$a_T = \nu' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = k\nu^2$ = curvature · (speed)^2

For computational purposes, we use

$$a_T = \frac{\vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} \quad \text{and} \quad a_T = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|}$$

Example:

$$\vec{\boldsymbol{r}}(t) = < \cos(t), \sin(t), t >$$

Find the tangential and normal components of acceleration.